Stability and Efficiency in the General Priority-based Assignment*

Preliminary Draft

Aytek Erdil† Taro Kumano‡

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Abstract

A school choice problem is a priority-based assignment problem. A priority ranking of a school in practice inherently exhibits indifference relations among the subsets of the set of students. We introduce a general class of priority rankings over sets of students, which captures both indifferences and substitutability. Our notion of substitutability ensures the existence of stable assignments. The characterization of efficient priority structures implies that there is usually a conflict between efficiency and stability. Thus we turn to the problem of finding a constrained efficient assignment, and give an algorithm which solves the problem for any priority structure that falls in our class. In an important application, school priorities that care about affirmative action can be captured by our model, but not previous models in the literature.

Keywords: substitutability with ties, stable improvement cycle, priority-based assignment, Pareto efficiency, acyclicity, non-wastefulness, school choice.

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†University of Cambridge, Faculty of Economics, Sidgwick Avenue, Cambridge, CB3 9DD, UK. Email: nae25@cam.ac.uk

‡Washington University in St. Louis, Department of Economics, One Brookings Drive, St. Louis, MO 63130, USA. E-mail: tkumano@wustl.edu

1 Introduction

A popular and widely used school admissions practice is to allocate school seats taking into account student preferences. Though such policies are often called *school choice*, obvious scarcity constraints arise due to some schools being more demanded than others. Therefore a well-defined procedure is necessary to decide how the over-demanded schools are assigned. Abdulkadiroğlu and Sönmez (2003) formulated these concerns in a natural and appealing way, and their approach has since been at the heart of various school choice programs. Each school is endowed with a certain number of seats and an exogenous priority order over the set of students. For a matching of students to schools to be *stable*, a student i must not be left envying another student j at school x, while i has higher priority for x than j. If these priorities are strict orders and interpreted as schools' preferences over students, then we are back in Gale and Shapley's (1962) college admissions model.

For each school, however, there are several concerns that guide such decisions: (1) Can *siblings* attend the same school? (2) Do pupils get schools within their *walk-zone*? (3) Does the procedure *treat otherwise-equal students equally*? (4) Is the student body at a given school *diverse* (gender equality or racial balance)? For a socio-economic perspective, the Office of Educational Research and Improvement explicitly suggests that an assignment should be based on those concerns, and in the same time the question arises what kind of priorities can reflect such concerns. To our best knowledge, concerns only for (1), (2) and (3) can be captured by the priority rankings over the set of students which allow ties among them, but a concern for diversity is different from the other three in that it cannot be necessarily generated by the ranking over the set of students, and it inherently exhibits indifference relations between some of the subsets of the set of students. Our challenge is to deal with all those concerns in an unified framework.

Let's consider the following example: there are six students who want to enter a school but it has just two seats. Six students are consisted of two white, two black, and two asian students, and the school wants to make a class as racially diverse as possible. In this case, every pair of students with different race should be the most preferred, and any of those pairs is equally preferable. It is clear that the ranking of the school inherently exhibits indifference relations among every pair of students with different race, and is over the subsets of the set of students. Note that this kind of priority ranking cannot be generated by a priority ranking over the students because the school's priority ranking is affected by whom you make a pair with.[1]

In this paper, we introduce a general class of priority rankings over the subsets of the set of students, *substitutable priorities with ties*, which captures indifferences and substitutability. We are able to study natural and appealing priority structures which take into account both *diversity* and *equal treatment*. We show the existence of stable matchings via a modification of the deferred acceptance algorithm. As in Erdil and Ergin (2008), there is a multiplicity of constrained efficient assignments, and arbitrarily breaking the ties can lead to constrained inefficiency. If the latter happens, the stable improvement cycles would improve upon the assignment to finally return a constrained efficient matching.

While a priority order over the students does not automatically lead to a ranking over sets of students, there is a simple class of preferences over sets studied widely in the literature. In a large class of two-sided matching models (see, e.g., Roth and Sotomayor, 1990), a ranking over sets is assumed to be *responsive* to a strict preference ranking over individuals. This condition essentially says that for any two sets that differ in only one student, the set containing the student with higher priority is ranked higher.[2] Then, core-stability is equivalent to pairwise stability. The existence of a stable matching is ensured, and Gale and Shapley's deferred acceptance algorithm gives the student optimal stable assignment. Erdil and Ergin (2008) begins with a model in which priorities can have ties, because many school choice programs declare large classes of students to be of equal priority. Gale and Shapley's algorithm does not necessarily give a student optimal assignment, but the *stable improvement cycles* always takes us to a constrained efficient outcome. Though the model with ties captures "equal treatment", it is not possible to "prioritize diversity" via a responsive priority order. If the priorities are responsive, then the ranking between sets $S \cup \{i\}$ and $S \cup \{j\}$ should not depend on what S is.[3]

Kelso and Crawford (1983) introduced a class of rankings over sets significantly larger

[1] Typically, the priorities are far from strict. For a given school many students are of *equal priority*, which is captured by a model which allows indifferences. See, e.g., Erdil and Ergin (2008).

[2] Formally, a ranking $\succsim$ over the sets of students is said to be *responsive* to a ranking $\succsim'$ over the set of students if whenever $i \succsim' j$, we have $\{i\} \cup S \succsim \{j\} \cup S$ for any S.

[3] An explicit example is in the next section.

than that of responsive rankings. Their generalization of the Gale-Shapley process, *the salary adjustment process* ensures that if firms' preferences over sets of workers satisfy the *gross substitutes* condition, then the core of the matching market is non-empty. Related to a diversity concern, Abdulkadiroğlu (2005) formulates priority rankings in a controlled school choice problem. Roughly speaking, the part of a school's seats are reserved for a specific type of students, and he shows that priority rankings respecting type specific quotas fall in substitutable priorities. We discuss his formulation and ours carefully in the next section.

The rest of the paper is organized as follows: we see the leading example in Section 2, Section 3 introduces a model, we discuss stability and efficiency in Section 4, we consider the way to find a constrained efficient assignment in Section 5, we demonstrate our class of priorities in a controlled school choice setting in Section 6.

2 Motivating Example

Prior to ours, Abdulkadiroğlu (2005) considers a school choice problem with a diversity concern in a priority-based assignment problem. In this model, each student is endowed with a certain type, there are type specific quotas for each school and it prioritizes the subsets of the set of students by two rules. If the subset of the set of students satisfies a type specific quota constraint, then they are considered to be acceptable, otherwise unacceptable. Among acceptable subsets of the set of students, they are ranked responsive to a ranking over the set of students.

Recall the example in the introduction: there are six students, $N = W \sqcup B \sqcup A$ where $W = \{w_1, w_2\}$, $B = \{b_1, b_2\}$ and $A = \{a_1, a_2\}$, and there is no exogenous priority order over them. Apart from a race equality policy, they are to be "treated equally". The approach of Abdulkadiroğlu (2005) leads to the following priority ranking:

$$\begin{aligned} \{w_1, b_1\} &\succ \{w_1, a_1\} \succ \{w_1, b_2\} \succ \{w_1, a_2\} \succ \{b_1, w_2\} \succ \cdots \\ &\succ \emptyset \\ &\succ \underbrace{\{w_1, w_2\}, \{b_1, b_2\}, \{a_1, a_2\}}_{\text{they do not satisfy the constraint}} \end{aligned}$$

Note that if all students are of equal priority, $\{w_1, b_1\}$ and $\{w_1, a_1\}$ should be treated equally in light of a diversity concern. However, the above formulation results in a biased assignment, that is, as long as $\{w_1, b_1\}$ are included in applicants a school never chooses any other pair of students with different race. In this case, white and black students are thought of as having a higher priority than any asian students. More importantly, an assignment produced by the above ranking may end up with being wasteful. Suppose only w_1 and w_2 apply for this school. Even though there are enough seats available, only one of them can be admitted.

On the other hand, a natural and desirable priority ranking may be as follows:

$$\{w_i, b_j\} \sim \{w_i, a_k\} \sim \{b_j, a_k\} \succ \{w_1, w_2\} \sim \{b_1, b_2\} \sim \{a_1, a_2\} \succ \cdots , \qquad (\star)$$

where $i, j, k \in \{1, 2\}$. As far as we know this priority order does not fit into any model previously studied in the literature.

First of all, it allows ties between sets of students. The earlier models which studied ties in priority orders begin with a weak order (i.e., a complete, transitive binary relation) on the set of students. Then they impose that the priority order on sets is responsive to that weak order on students. It is not hard not see that the above priority order $\succsim$ cannot be generated in that fashion. If $\succsim$ were responsive to an order $\succsim'$ on $\{w_1, w_2, b_1, b_2, a_1, a_2\}$, we would have

$$\{w_1, b_1\} \succ \{w_1, w_2\} \quad \Longrightarrow \quad b_1 \succ' w_2$$

and

$$\{w_2, b_2\} \succ \{b_1, b_2\} \quad \Longrightarrow \quad w_2 \succ' b_1$$

a contradiction.

In addition, when all students apply to a school, the allocations obtained by the above priority structure cannot be generated by splitting a school into sub-schools: each sub-school prioritizes students with one race over those with the other, since there are only two seats and the number of race is three, so that there should be students with some race who are always ranked lower. This kind of treatment is sometimes called a type specific quota in the previous literature.

3 Preliminaries

Let N be a set of students, and X a set of schools. There are q_x seats at school x, for $x \in X$. These schools are to be assigned to the students such that each student receives at most one seat, and the allocation has to *respect exogeneously given priorities*, a notion formalized below.

Each school has a priority ranking $\succsim_x$ over all subsets of N. Formally speaking, $\succsim_x$ is a complete, transitive binary relation over 2^N. A **priority structure** $\succsim$, is a vector of priority orders $(\succsim_x)_{x \in X}$. If $S \succsim_x T$ and $T \succsim_x S$, then we write $S \sim_x T$, and say S and T are of equal priority with respect to $\succsim_x$. Clearly, being of equal priority is an equivalence relation. If $S \succsim_x T$, but not $T \succsim_x S$, then we write $S \succ_x T$.

When the demand for a school exceeds the supply, one can appeal to priorities to decide which students are to be assigned to the school. In other words, given a school x with q_x seats and a priority order $\succsim_x$, we have a choice correspondence $\mathcal{C}_x : 2^N \rightrightarrows 2^N$ such that

$$S' \subseteq S \text{ and } |S'| \leq q_x \text{ for each } S' \in \mathcal{C}_x(S)$$
$$S' \in \mathcal{C}_x(S) \iff S' \succsim_x S'' \text{ for all } S'' \subseteq S \text{ with } |S''| \leq q_x$$

Given a priority rule $\mathcal{C}_x$ and a set S, we define the set of **definitely chosen students**

$$DC_x(S) \;=\; \bigcap_{S' \in \mathcal{C}_x(S)} S' \;=\; \{i \in S \mid i \in S' \text{ for all } S' \in \mathcal{C}_x(S)\}$$

Note that $DC_x(S)$ can be empty.

Definition 1 A priority structure is **substitutable** if for each $x \in X$, for each $S, T \subseteq N$ with $S \subseteq T$,

(a) for each $T' \in \mathcal{C}_x(T)$, we have $T' \cap S \subseteq S'$ for some $S' \in \mathcal{C}_x(S)$.

(b) for each $S' \in \mathcal{C}_x(S)$, we have $T' \cap S \subseteq S'$ for some $T' \in \mathcal{C}_x(T)$.[4]

[4] It is helpful to define the **rejection correspondence** $\mathcal{R}_x$, which associates to each $S \subseteq N$, the family of subsets of S which can be rejected from among S. That is,

$$\mathcal{R}_x(S) = \{T \subseteq S \mid T = S \backslash S' \text{ for some } S' \in \mathcal{C}_x(S)\}.$$

The condition (b) of Definition 1 can be rewritten as

(b') for each $S' \in \mathcal{R}_x(S)$, we have $S' \subseteq T'$ for some $T' \in \mathcal{R}_x(T)$.

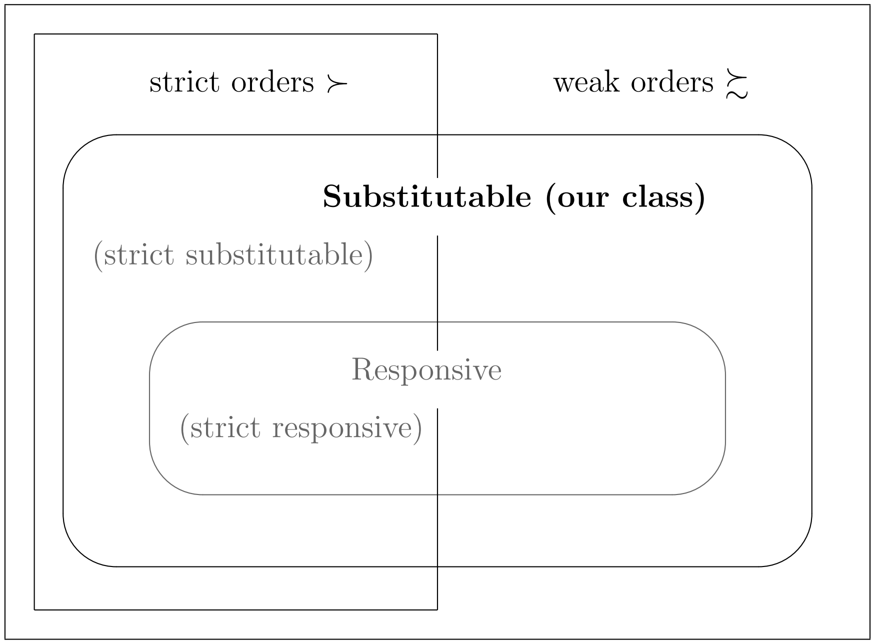

Figure 1: Our class includes substitutability in strict sense and weak responsiveness.

This definition covers various environments studied in the literature. For example, responsive preferences over sets of doctors (Roth, 1984), the school choice formulation of Abdulkadiroğlu and Sönmez (2003), and the school priorities with ties (e.g., Erdil and Ergin, 2008) are all special cases.[5] One contribution of this paper is that our generalization goes beyond, and covers natural priority structures which are not captured by any of the aforementioned models:

Example 1 (Race equality) Remember the following priority structure is considered to be desirable in the previous sections. We see that it in fact falls in our class.

$$\{w_i, b_j\} \sim \{w_i, a_k\} \sim \{b_j, a_k\} \succ \{w_1, w_2\} \sim \{b_1, b_2\} \sim \{a_1, a_2\} \succ \cdots ,$$

[5]Our definition is easily extended to allow different contracts between a student and a school. Then each school has a *weak* ranking over sets of contracts, and we get a generalization of Kelso and Crawford (1982) and Hatfield and Milgrom (2005).

Note that if $\mathcal{C}_x$ is a function for each $x \in X$, then the conditions (a) and (b) in Definition 1 are equivalent. While Kelso and Crawford (1982) use the formulation (a), Hatfield and Milgrom (2005) use the formulation (b'). In our generalized environment, these conditions do not imply each other any more.

where $i, j, k \in \{1, 2\}$.

We only need to check cases where the size of a smaller set (a counterpart of S in the definition) is at least as large as two. If S consists of students with different race, then only pairs of students with different race are chosen in S, and in any larger set they should also be chosen. Note that no pair of students with the same race is chosen in both sets. Hence the conditions (a) and (b) hold. Otherwise, S consists of two students with the same race. The condition (b) automatically holds because there is no student rejected. In a larger set, there should be a student with different race from S and any pair of students with different race will be chosen. Therefore, any intersection of a pair of students chosen in a larger set and S must be one of S and not both. Clearly, the condition (a) is also satisfied.

For the allocation problem, what we really need is a choice correspondence. In the above example, the ranking over singletons does not matter as long as they are acceptable.[6] $\Diamond$

Let μ be an assignment such that $\mu : N \to X \cup N$ with the following properties:

$$\forall i \in N, \qquad \mu(i) \in X \cup \{i\}$$
$$\forall x \in X, \qquad |\mu^{-1}(x)| \leq q_x$$

Each student i has a strict preference ranking R_i over $X \cup \{i\}$, where receiving i is interpreted as getting one's outside option. P_i denotes the strict part of R_i. Given a preference profile $R = (R_i)_{i \in N}$, we have a Pareto domination relation over all possible allocations.

Definition 2 Given students' preferences R, an assignment μ **respects priorities** $\succsim$, if for each $i \in N$, $\mu(i) R_i i$, and for each $x \in X$, we have $\mu^{-1}(x) \in \mathcal{C}_x(\{i \mid x R_i \mu(i)\})$.

The definition captures the idea that there should not be a set S more deserving of the school x, than those students currently assigned.

[6] If $q_x > 1$, then a priority order $\mathcal{C}_x$ does not necessarily provide a comparison between singletons. For example, let $q_x = 2$, and consider $\mathcal{C}_x$ such that

$$\mathcal{C}_x(\{1, 2, 3, 4\}) = \{\{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}\},$$

whereas for any S with $|S| = 3$, all two-element-subsets of S are chosen. This priority order cannot be generated from a weak order over the set of students using responsiveness.

Recall that a matching is called **pairwise stable** if it is not blocked by an individual student, or an individual school, or a student-school pair. That is, (1) each student i prefers her match to being unassigned; (2) each school prefers not to get rid of some of the assigned students; and (3) there is no student-school pair who are not matched, but would rather be matched. Formally,

(1) For all $i \in N, \quad \mu(i) R_i i$

(2) For all $x \in X, \quad \mu^{-1}(x) \in \mathcal{C}_x(\mu^{-1}(x))$

(3) There is no $(i, x) \in N \times X$ such that $x P_i \mu(i)$ and $i \in DC(\mu^{-1}(x) \cup \{i\})$

Clearly, respecting priorities implies pairwise stability. When does the converse hold? The next proposition gives an answer.

Proposition 1 *Let $\succsim$ be a substitutable priority structure. Then an assignment is pairwise stable if and only if it respects priorities. In other words, the set of pairwise stable assignments is the same as the weak core.*

From now on, we say an assignment which respects priorities a *stable* assignment. Does there always exist a stable assignment? A natural extension of Gale and Shapley's Deferred Acceptance Algorithm shows constructively that it does if the priority structure is substitutable.

Modified Deferred Acceptance Algorithm (MDA)

Round 1: All students apply to their favorite schools. For each school x, if A_x^1 is the set of applicants, *an* element S_x^1 in $\mathcal{C}_x(A_x^1)$ is declared the temporary winners, and the rest of the applicants, denoted $Z_x^1 = A_x^1 \backslash S_x^1$ are rejected.

⋮

Round t: Those who were rejected in round $t-1$, apply to their next favorite school. For each school x, if A_x^t is the set of all students who have applied to x so far, *a* set of temporary winners $S_x^t \in \mathcal{C}_x(A_x^t)$ is chosen such that $Z_x^{t-1} \subseteq A_x^t \backslash S_x^t$.

[This ensures that those students that were rejected by x in a previous round are still rejected.]

When every student is either matched with a school or has been rejected by all schools in his list, the algorithm ends.

Proposition 2 *Given a substitutable priority structure $\succsim$, the Modified Deferred Acceptance Algorithm returns a stable assignment.*

The above algorithm is a generalization of the student-proposing deferred acceptance algorithm to an environment which allows school priority rankings over sets of students to be substitutable with ties. Note that "monotonicity of the rejection correspondence", i.e., the condition (b) of substitutability is enough for the proposition 2 if an assignment the MDA returns is pairwise stable, however, without the condition (a) of substitutability, we might have a pairwise stable outcome which does not respect priorities.

Example 2 Suppose there are two schools, $\{x, y\}$, and five students, $\{i_1, i_2, i_3, i_4, i_5\}$. Students' preferences are:

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
x	y	x	y	x
		y	x	

And the priority structures are:

$\mathcal{C}_x$	$\mathcal{C}_y$
$\{i_1, i_2\}, \{i_3, i_4\}$	$\{i_2, i_3\}, \dots$
$\{i_1, i_3\}, \{i_1, i_4\}, \{i_1, i_5\}, \{i_2, i_3\}, \{i_2, i_4\}$	$\{i_2, i_4\}, \dots$
$\{i_2, i_5\}, \{i_3, i_5\}, \{i_4, i_5\}$	

This priority structure satisfies (b) but not (a). The MDA gives

1. Students i_1, i_3 and i_5 apply to a school x, and $\{i_1, i_5\}$ can be chosen. A student i_3 is rejected. A school y tentatively holds $\{i_2, i_4\}$.

2. A student i_3 applies to y, and she is tentatively accepted. A student i_4 is rejected.

3. A student i_4 applies to x, but a school x can hold $\{i_1, i_5\}$, and she is rejected. The algorithm ends.

The assignment is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & y & y & \emptyset & x \end{pmatrix}.$$

This is pairwise stable but does not respect priorities since both students 3 and 4 prefer x rather than their assigned schools, and the school x prefers $\{i_3, i_4\}$ to $\{i_1, i_5\}$. $\diamondsuit$

Definition 3 A priority structure $\succsim$ is **acceptant** if for each $x \in X$, for each $S \subseteq N$, and for each $S' \in \mathcal{C}_x(S)$ we have $|S'| = \min\{|S|, q_x\}$.

This captures the idea that an unused school seat cannot be denied to any student. No-blocking-pairs (NB) condition together with an acceptant priority structure means that stability implies **non-wastefulness**, i.e., if there exists $i \in N$ such that $aP_i\mu(i)$, then $|\mu^{-1}(x)| = q_x$.

As our model generalizes the one studied in Erdil and Ergin (2008), it follows that: (1) The outcome of the generalized deferred acceptance algorithm is not necessarily constrained efficient. (2) The constrained efficient set is not necessarily a singleton.

Given $\succsim$, define the **constrained efficient correspondence** $f^{\succsim}$, which assigns to each preference profile R, the set of stable assignments which are not Pareto dominated by another stable assignment.

4 Stability vs. efficiency

Let us call a priority structure $\succsim$ **efficient** if $f^{\succsim}$ is Pareto efficient. Ergin (2002) characterizes the efficient priority structures under the assumption of responsive priorities without ties. Ehlers and Erdil (2010) give a more general characterization allowing for ties in priority orders. Below, we will let the priorities be acceptant substitutable with ties, providing the most general statement in a much larger environment. This characterization result, as the ones before, confirms that $f^{\succsim}$ is Pareto efficient under very restrictive conditions.

Definition 4 Given a priority structure $\succsim$, a **weak cycle** is constituted of distinct $i, j, k \in N$, and $x, y \in X$ such that there exist $S_x, S_y \subseteq N \backslash \{i, j, k\}$ with $S_x \cap S_y = \varnothing$

satisfying

$$
\textbf{(C)} \quad \begin{array}{l} j \notin DC_x(S_x \cup \{i, j\}) \\ j \in DC_x(S_x \cup \{k, j\}) \\ k \notin DC_x(S_x \cup \{i, k\}) \\ i \notin DC_y(S_y \cup \{k, i\}) \end{array}
$$

$$
\textbf{(S)} \quad |S_x| = q_x - 1 \text{ and } |S_y| = q_y - 1.
$$

If $\succsim$ does not have any weak cycle, then it is called **strongly acyclic**.

Proposition 3 *Let $\succsim$ be an acceptant substitutable priority structure. $f^{\succsim}$ is efficient if and only if $\succsim$ is strongly acyclic.*

Ergin (2002) characterizes acyclicity as similarity of priorities. However, our acyclicity requires a much tighter interpretation of priorities.

Example 3 Recall the example of race equality. Students are $N = \{w_1, w_2, b_1, b_2, a_1, a_2\}$. Suppose there are two schools x, y, each with two seats. Consider both schools x and y have exactly the same priority structure as in the example.

$$
\{w_i, b_j\} \sim \{w_i, a_k\} \sim \{b_j, a_k\} \succ \{w_1, w_2\} \sim \{b_1, b_2\} \sim \{a_1, a_2\} \succ \cdots ,
$$

We see that it is weakly cyclic. Let $S_x = \{a_2\}$ and $S_y = \{b_2\}$. (S) holds.

$$
\begin{aligned}
\mathcal{C}_x(\{a_2, \underbrace{w_1}_{i}, \underbrace{b_1}_{j}\}) &= \{\{a_2, w_1\}, \{a_2, b_1\}, \{w_1, b_1\}\} \\
\mathcal{C}_x(\{a_2, b_1, \underbrace{a_1}_{k}\}) &= \{\{b_1, a_1\}, \{b_1, a_2\}\} \\
\mathcal{C}_x(\{a_2, w_1, a_1\}) &= \{\{w_1, a_1\}, \{w_1, a_2\}\} \\
\mathcal{C}_y(\{b_2, w_1, a_1\}) &= \{\{w_1, b_2\}, \{w_1, a_1\}, \{b_2, a_1\}\}
\end{aligned}
$$

which implies

$$
\begin{array}{l} b_1 \notin DC_x(S_x \cup \{w_1, b_1\}) \\ b_1 \in DC_x(S_x \cup \{a_1, b_1\}) \\ a_1 \notin DC_x(S_x \cup \{w_1, a_1\}) \\ w_1 \notin DC_y(S_y \cup \{a_1, w_1\}). \end{array}
$$

Note that when preferences are

R_{w_1}	R_{w_2}	R_{b_1}	R_{b_2}	R_{a_1}	R_{a_2}
y		x	y	x	x
x				y	

$\mu = \begin{pmatrix} w_1 & w_2 & b_1 & b_2 & a_1 & a_2 \\ x & w_2 & b_1 & y & y & x \end{pmatrix}$ is constrained efficient, but not efficient. $\Diamond$

5 Stability Preserving Pareto Improvement

In the absence of ties, we know that $f^{\succsim}$ is singleton valued and reached simply by the deferred acceptance algorithm. On the other hand, when there are ties, the constrained efficient correspondence is not necessarily singleton-valued. Moreover, arbitrarily breaking the ties as we execute the deferred acceptance algorithm may lead to constrained inefficiency. In the case of responsive priorities the *stable improvement cycles algorithm* by Erdil and Ergin (2008) reaches a constrained efficient assignment. We explore whether such cycles can be used to solve the similar problem when priorities are acceptant and substitutable.

A special case of our environment is that of responsive priorities with ties. Motivated by the fact that an arbitrary resolution of ties in implementing the DA algorithm may lead to an assignment which is not constrained efficient, Erdil and Ergin (2008) explored stability preserving Pareto improvements. A *stable improvement cycle* is a cycle of distinct schools such that for any edge $x \to y$, there is a student i_x matched with x, who would like to be matched with y instead, and is one of the highest y-priority students among those who would like to move to y. They show that if a stable assignment is not constrained efficient, then it must admit a stable improvement cycle, and therefore by simply searching for stable improving cycles and implementing them successively, one can reach a constrained efficient assignment.

In our more general environment, Erdil and Ergin's definition does *not* capture all the improvement cycles that preserve stability. That is, it is possible that a stable matching μ is Pareto dominated by another stable matching ν, and μ does not admit a stable improvement cycle in the sense of Erdil and Ergin (2008).[7] This is because in

[7] An explicit example is in Remarks.

our environment, the priority of a student is not absolute but relative in a sense that it depends on whom his colleagues are. Hence, we need to take it into account.

Given a stable assignment μ, who could be replacing, without violating stability, j's position at $\mu(j)$ if j were to disappear? It must be that when such a student ℓ replaces j, the new set of students must be chosen set in the face of those who would like to be replacing j at $\mu(j)$. To formalize this idea in general, let $\mathcal{E}_j^\mu$ stand for the set of students who envy j at assignment μ:

$$\mathcal{E}_j^\mu = \{i \mid \mu(j) P_i \mu(i)\}.$$

Then, the set of **students who can replace student j at μ** is

$$E_j^\mu = \{\ell \mid \{\ell\} \cup \mu^{-1}(\mu(j)) \backslash \{j\} \in \mathcal{C}_{\mu(j)}(\mathcal{E}_j^\mu \cup \mu^{-1}(\mu(j)) \backslash \{j\})\}$$

Note that each E_j^μ is not necessarily singleton and any two E_j^μ and $E_{j'}^\mu$ are not necessarily distinct. If we find a cycle of students, then it is feasible to exchange their assignments, and formally,

Definition 5 Given a priority structure $\succsim$, a preference profile R and a stable assignment μ, a **stable student improving cycle** consists of distinct students $i_0, i_1, \ldots, i_{n-1}, i_n = i_0$ such that $i_\ell \in E_{i_{\ell+1}}^\mu$ for all $\ell = 0, \ldots, n-1$.

The first relation between a constrained efficient assignment and a stable student improving cycle is the following:

Proposition 4 *Given an acceptant and substitutable priority structure $\succsim$, if a stable assignment does not admit a stable student improving cycle then it is constrained efficient.*

Unfortunately, the converse does not hold in general.[8] We further impose a weak assumption on the priority structure to hold the converse.

Suppose that a priority structure $\succsim$ is acceptant substitutable. We will define a weak form of "equal treatment of equals" as follows:

[8] An explicit example is in Remarks

Definition 6 Let us say that $\succsim$ satisfies **equal treatment of equal students** if given $\{i,j\} \cup S \subseteq T$, and $\{i,j\} \cup S' \subseteq T'$ such that $T \subseteq T'$, $|S| = q_x - 1$ and $|S'| = q_x - 1$,

$$S \cup \{i\}, S \cup \{j\} \in \mathcal{C}_x(T) \text{ and } S' \cup \{i\} \in \mathcal{C}_x(T') \Longrightarrow S' \cup \{j\} \in \mathcal{C}_x(T'). \qquad \textbf{(ETE)}$$

Which students are to be treated equally can change from one school to the other.[9] However, the critical requirement is that if two students are of equal priority in applicants at some school, then one can be replaced with the other in any larger applicants.

Proposition 5 *Assume an acceptant and substitutable priority structure $\succsim$ satisfies ETE. Suppose that the stable assignment μ admits a SIC, and let the shortest SIC be*

$$i_0 \to i_1 \to \cdots \to i_{n-1} \to i_0.$$

If the assignment ν is obtained by carrying out this cycle, i.e., if

$$\nu(i) = \begin{cases} \mu(i_{\ell+1}) & \text{if } i = i_\ell \\ \mu(i) & \text{otherwise} \end{cases},$$

then ν is stable.

Corollary 1 *Whenever an acceptant and substitutable priority structure $\succsim$ satisfies ETE, a stable assignment is constrained efficient if and only if it does not admit a stable student improving cycle.*

5.1 An algorithm

The above proposition leads to an algorithm whose outcome is a constrained efficient assignment. Starting from a stable assignment μ, one needs to construct a graph whose set of vertices is the set of students. For any pair (i,j) of vertices, there will be an edge from i to j if and only if $i \in E_j^\mu$. If this graph does not have a cycle which preserves stability, then μ is constrained efficient. Otherwise, we can let the cycle lead to a Pareto improving cyclic trade which would preserve stability.

Step 0:

[9] Note that our prioritizing diversity as in Section 2 satisfies the above property of equal treatment.

Run the *Modified Deferred Acceptance Mechanism* to obtain an initial matching μ^0.

Step t $\geq$ 1:

$(t.a)$ Given μ^{t-1}, let the students stand for the vertices of a directed graph, where for each pair of students i and j, there is an edge $i \longrightarrow j$ if and only if $i \in E_j^{\mu^{t-1}}$.

$(t.b)$ If there are any stable student improving cycles in this directed graph, *select* a shortest one, and carry out this cycle to obtain μ^t, and go to step $(t+1.a)$. If there is no such cycle, then return μ^{t-1} as the outcome of the algorithm.

This algorithm will return a student optimal stable assignment, but when there are more than one such assignments, the particular outcome will depend on the selections in running the MDA in Step 0, and the specification of the cycle search in later steps.

On the one hand, the algorithm ensures a constrained efficient outcome. On other hand, we know from Proposition 3 that if $\succsim$ is acyclical, then any constrained efficient assignment is Pareto efficient. Thus we have

Corollary 2 *If $\succsim$ is acyclical, then the above algorithm Pareto efficient.*

6 Application

We demonstrate a natural and practical subclass of our substitutability, called *a priority respecting type-specific quotas*. For a socio-economic purpose, a school usually sets a part of its seats for a specific characteristic, such as students with disability or in minority. For example, when a school wants to make a class racially balanced, this concern is reflected by splitting a whole seats into each race. The question is whether such a socio-economic policy is effective or not, in the sense that each prioritized student is better off by a policy or not.

Kojima (2010) argues that such a policy does not necessarily achieves that purpose. He gives the case that even though some group is given a good treatment by setting a specific quota, every student in the group is worse off, compared to the assignment without treatment. However, we are anxious about the result since comparison seems to be made by two different situations, wasteful and non-wasteful priorities.

In this section, we assume that each school initially has a priority ranking over students, and if it does not employ any specific policy, we assume that its priority follows responsiveness. Otherwise, depending on its type-specific quotas, we offer a way of constructing a priority structure over the sets, which both reflects an initial ranking and respects type-specific quotas.

A pre-priority, which is a weak order over the students, is denoted by $\succsim^{pre} \in N \times N$. (A strict part is denoted by $\succ^{pre}$, and indifference is denoted by $\sim^{pre}$.) Let a type space $T = \{\tau_1, \dots, \tau_n\}$. Each student is in one of types, and a type function, $\tau : N \to T$, indicates it. For every school x, there are type-specific quotas $q_x^T = (q_x^{\tau_1}, \dots, q_x^{\tau_n})$ such that $1 \leq q_x^\tau \leq q_x$, and $\sum_\tau q_x^\tau = q_x$. S_τ denotes $\{s \in S : \tau(s) = \tau\}$.

The set of students is classified into the following families. $\forall S \subseteq N$ with $|S| = q_x$,

$$
\begin{aligned}
D_0 &= \left\{ S \subseteq N : \sum_{\tau \in T} ||S_\tau| - q^\tau| = 0 \right\} \\
D_1 &= \left\{ S \subseteq N : \sum_{\tau \in T} ||S_\tau| - q^\tau| = 2 \right\} \\
&\vdots \\
D_a &= \left\{ S \subseteq N : \sum_{\tau \in T} ||S_\tau| - q^\tau| = 2a \right\} \\
&\vdots
\end{aligned}
$$

Definition 7 $\succsim$ *satisfies a Respecting Constraint (RC) if*

$$S' \in D_a \text{ and } S'' \in D_{a+1} \Rightarrow S' \succ S''$$

Definition 8 $\succsim$ *satisfies a Restricted Responsiveness (RR) if for every* $T \cup \{s'\}$, $T \cup \{s''\} \in D_a$, *we have*

$$T \cup \{s'\} \succsim T \cup \{s''\} \Leftrightarrow s' \succsim^{pre} s''$$

We say that the priority structure respects type-specific quotas if a priority structure constructed from a pre-priority satisfying (RC) and (RR). These notions are similar to Abdulkadiroğlu (2005) but not a generalization. Our definition not only captures indifferences but also is compatible with non-wastefulness, but Abdulkadiroğlu (2005) is not. The following claim shows that an acceptant priority satisfying (RC) and (RR) falls in our class.

Claim 1 *For every pre-priority $\succsim^{pre}$, a priority structure $\succsim$ constructed from $\succsim^{pre}$ satisfying (RC) and (RR) is acceptant and substitutable.*

Therefore, when schools employ a priority respecting type-specific quotas, we know that the MDA always returns the stable assignment. Furthermore, we can directly show the following:

Claim 2 *A stable assignment is constrained efficient if and only if it does not admit a stable student improving cycle.*

Remark that a SIC is not consisted of distinct schools, and we do not apply a stable improving cycle in Erdil and Ergin (2008).

Example 4 (Racial Balance & Walk Zone) Three quotas, $(q^w, q^b, q^a) = (1, 1, 1)$. Students are pre-prioritized by within walk zone and out of walk zone, as follows:

$$\{w_1\} \sim^{pre} \{w_2\} \sim^{pre} \{b_1\} \succ^{pre} \{b_2\} \sim^{pre} \{a_1\}$$

A priority respecting type-specific quotas is

$$\begin{aligned}
&\{w_1, b_1, a_1\} \sim \{w_2, b_1, a_1\} \succ \\
&\{w_1, b_2, a_1\} \sim \{w_2, b_2, a_1\} \succ \\
&\{w_1, w_2, b_1\} \succ \\
&\{w_1, w_2, b_2\} \sim \{w_1, b_1, b_2\} \sim \{w_1, w_2, a_1\} \succ \\
&\{b_1, b_2, a_1\} \succ \dots
\end{aligned}$$

Clearly this is not responsive. This example is different from one about gender equality in that an underlining priority is not necessarily indifferent. $\Diamond$

In Kojima (2010), the implication of introducing the affirmative action policy is discussed. He changes the notion of stability in that an assignment is "stable" if it is feasible, individually rational, and there is no blocking pair within feasible assignments.

The main result (Theorem 1) is briefly that there is no stable mechanism that an increase of the number of some type-specific quota makes agents in that type weakly better off. The proof is by example, and the following example grasps the point.

Example 5 (Compared to the previous literature) Suppose there are two schools $\{x, y\}$ and both have 2 seats. There are 4 students, denoted by m_1, m_2, w_1, w_2. Preferences and pre-priorities are as follows:

R_{m_1}	R_{m_2}	R_{w_1}	R_{w_2}
x	x	y	y
y		x	

and

$\succsim_x^{pre}$	$\succsim_y^{pre}$
m_1	m_1
m_2	m_2
w_1	w_1
w_2	w_2

(1) Suppose there is no policy. Then $\succsim_x$ and $\succsim_y$ are responsive to pre-priorities, and the stable outcome by the DA is just

$$\mu = \begin{pmatrix} m_1 & m_2 & w_1 & w_2 \\ x & x & y & y \end{pmatrix}$$

(2) Suppose both schools have a gender equal policy. Then $\{m_1, m_2\}$ and $\{w_1, w_2\}$ are unacceptable in Abdulkadiroğlu (2005) or infeasible in Kojima (2010).

Then the DA outcome is

$$\mu' = \begin{pmatrix} m_1 & m_2 & w_1 & w_2 \\ x & \emptyset & y & \emptyset \end{pmatrix}$$

If stability is re-defined as the outcome which does not allow individual or a pairwise deviation within the feasible outcomes, then the above assignment is “stable” in this sense. You can see that compared with μ, m_2, w_2 are worse off, even though a gender equal policy is employed.

We want to point out here that it comes from wastefulness, but from any specific policy. Even though there is a vacancy in both schools, they leave it vacant in this formulation. Even though no woman wants a position at a school x, a gender equal constraint prohibits m_2 from attending a school x.

(3) Alternatively, if both schools construct a priority respecting type-specific quotas $(q_x^m, q_x^w) = (q_y^m, q_y^w) = (1, 1)$, and run the DA, then the outcome is trivially

$$\mu'' = \begin{pmatrix} m_1 & m_2 & w_1 & w_2 \\ x & x & y & y \end{pmatrix} = \mu$$

Note that both schools respect gender equality, but if there is no man-woman pair in applicants, then they accept the second best applicants. Furthermore, the notion of stability does not change. ♢

Remark that non-wastefulness and the formulation of Abdulkadiroğlu (2005) or Kojima (2010) are incompatible.

7 Discussion

In this paper, we develop a general class of priority rankings which captures indifferences and substitutability. As our example shows, a complicated but practical concern is well captured. In this section, we discuss how appropriate our class of priorities is and its limitation.

As we restrict attention to substitutability, it is because when priority structures are strict, Hatfield and Kojima (2008) show that substitutability is a maximal domain for the existence of stable assignments in a sense that

$$\succ \text{ is substitutable } \Leftrightarrow \forall R,\ S^{\succ}(R) \neq \emptyset,$$

where $S^{\succ}(\,\cdot\,)$ is the set of stable assignment under $\succ$. Since our class includes strict substitutable priorities as a special case, we do not go beyond substitutability.

This restriction excludes some interesting priority structure as follows:

Example 6 Suppose a school has two seats and there are four students, $\{s_1, s_2, s_3, s_4\}$, with the following attributes:

	s_1	s_2	s_3	s_4
race	black	black	white	white
gender	male	female	male	female

When a school respects both race and gender equality, a natural way of prioritizing them is

$$\{s_1, s_4\} \sim \{s_2, s_3\} \succ \{s_1, s_2\} \sim \{s_3, s_4\} \sim \{s_1, s_3\} \sim \{s_2, s_4\} \succ \cdots$$

This priority ranking does not satisfy the condition (a) in the definition 1, whereas the condition (b) holds. Suppose $S = \{s_1, s_2, s_3\}$ and $T = \{s_1, s_2, s_3, s_4\}$. Then $\mathcal{C}(T) = \{\{s_1, s_4\}, \{s_2, s_3\}\}$ but

$$\{s_1, s_4\} \cap S = \{s_1\} \not\subseteq \{s_2, s_3\} = \mathcal{C}(S).$$

Therefore, the condition (a) in the definition 1 does not hold. $\diamondsuit$

A critical difference from ours involves two or more dimensions of students' attributes. We may conclude that there is another conflict with stability, that is, stability and a general diversity concern.

A Proofs

A.1 Proof of Proposition 1

Denote $U = \mu^{-1}(x)$. If μ is pairwise stable, then it must be that for any ℓ with $xR_\ell\mu(\ell)$, we have $U \in \mathcal{C}_x(U \cup \{\ell\})$. Since students' preferences over schools are strict, we can write this in a seemingly stronger way: for any ℓ with $aR_\ell\mu(\ell)$, we have $U \in \mathcal{C}_x(U \cup \{\ell\})$. We would like to show that if $S \subseteq N$ such that $aR_i\mu(i)$ for all $i \in S$, then $U \in \mathcal{C}_x(U \cup S)$. In order to conclude via induction on $|S|$, it is sufficient to show that

$$[U \in \mathcal{C}_x(U \cup S) \text{ and } U \in \mathcal{C}_x(U \cup \{k\})] \quad \Rightarrow \quad U \in \mathcal{C}_x(U \cup S \cup \{k\}).$$

First, note that $\mathcal{C}_x$ being consistent with $\succsim_x$ implies

$$U \succsim_x V \quad \text{for all } V \subseteq U \cup S \text{ such that } |V| \leq q_x, \tag{$\star$}$$

and

$$U \succsim_x V \quad \text{for all } V \subseteq U \cup \{k\} \text{ such that } |V| \leq q_x,$$

Now, suppose for a contradiction that $U \notin \mathcal{C}_x(U \cup S \cup \{k\})$. Then for any $T \in \mathcal{C}_x(U \cup S \cup \{k\})$, we have

$$T \succ_x U.$$

Combining this with the relationship ($\star$) above, we get $T \succ_x V$ for all $V \subseteq U \cup S$ such that $|V| \leq q_x$, and we conclude that $T \nsubseteq U \cup S$. Therefore, for any $T \in \mathcal{C}_x(U \cup S \cup \{k\})$, we must have $k \in T$. On the other hand, $U \in \mathcal{C}_x(U \cup \{k\})$ implies that $\{k\} \in \mathcal{R}_x(U \cup \{k\})$, which implies, due to substitutability, $\{k\} \subseteq (U \cup S \cup \{k\}) \backslash T$ for some $T \in \mathcal{C}_x(U \cup S \cup \{k\})$, yielding the desired contradiction. □

A.2 Proof of Proposition 2

A_x^t is the set of students who have applied to school x in some round $k \leq t$. Hence

$$A_x^1 \subseteq A_x^2 \subseteq \cdots$$

The algorithm requires that those students rejected in rounds $k \leq t-1$ would still be rejected if they were considered to be among the applicant in round t. This can be ensured thanks to $\mathcal{C}_x$ being substitutable, because $Z_x^{t-1} = A_x^{t-1} \backslash S_x'$ for some $S_x' \in \mathcal{C}_x(A_x^{t-1})$ and $A_x^{t-1} \subseteq A_x^t$ together imply that there exist $Z_x^t = A_x^t \backslash S_x''$ such that $Z_x^t \supseteq Z_x^{t-1}$ for some $S_x'' \in \mathcal{C}_x(A_x^t)$.

In order to see that the algorithm indeed ends, note that at any round if a student is not matched, then she applies to her next favorite school in the following round. Therefore, she either exhausts all her acceptable schools by going down all the way to the end of her preference list, or ends up being matched with some school.

Suppose that μ is the matching obtained as a result of the algorithm which ends in round m. Stability of μ basically means

$$\mu^{-1}(x) \in \mathcal{C}_x(\{i \mid xR_i\mu(i)\})$$

But of course those who weakly prefer x to their match under μ are either matched with x, or have applied to x at some round of the algorithm. Thus, we need

$$\mu^{-1}(x) \in \mathcal{C}_x(A_x^m),$$

which clearly holds by the fact that $A_x^m \backslash \mu^{-1}(x) = Z_x^m \in \mathcal{R}_x(A_x^m)$. □

A.3 Proof of Proposition 3

The main part in proving the proposition is ($\Leftarrow$), i.e., showing that a strongly acyclic $\succsim$ leads to efficient $f^{\succsim}$. We will prove this part in two steps.

Given a priority structure $\succsim$, a **generalized weak cycle of size** n is constituted of distinct schools $x_0, x_1, \ldots, x_{n-1} \in X$ and distinct students $j, i_0, i_1, \ldots, i_{n-1} \in N$ with $n \geq 2$ such that

(1) $x_\ell \neq x_{\ell+1}$ for $\ell \in \{0, 1, \ldots, n-1\}$ (with $x_n = x_0$),

(2) there exist mutually disjoint sets of students $S_{x_0}, \ldots, S_{x_{n-1}} \subseteq N \backslash \{j, i_0, i_1, \ldots, i_{n-1}\}$ such that

$$\textbf{(C)} \qquad \begin{aligned} & j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}) \\ & j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}) \\ & i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\ & i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\ & \quad \vdots \\ & i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}) \\ & i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\}) \end{aligned}$$

$$\textbf{(S)} \qquad |S_{x_\ell}| = q_{x_\ell} - 1 \text{ for } \ell = 0, 1, \ldots, n-1.$$

Step 1: If there exists a Pareto inefficient assignment $\mu \in f^{\succsim}(R)$, then $\succsim$ has a generalized weak cycle.

Proof of Step 1: Suppose that $\mu \in f^{\succsim}(R)$ is not Pareto efficient. Of all the Pareto improvements over μ, let ν be one which has the least number of students improving over μ. Denote by N' the set of students who are better off under ν compared with μ:

$$N' = \{i \mid \nu(i) P_i \mu(i)\}.$$

Denote by $\mathcal{E}_j^\mu$ the set of students who envy the student j under μ:

$$\mathcal{E}_j^\mu = \{\ell \in N \mid \mu(j) P_\ell \mu(\ell)\}.$$

Set $\mathcal{E}'_j$ to be the set of students in N' who envy j. That is,

$$\mathcal{E}'_j = \mathcal{E}_j^\mu \cap N' = \{\ell \in N' \mid \mu(j) P_\ell \mu(\ell)\}.$$

If $j \in N'$, then by the reshuffling lemma[10],we have $\mu(j) \in \nu(N')$. In particular, $\mu(j)$ is desired by some student in N' under μ, and hence $\mathcal{E}'_j$ is nonempty. Because μ respects priorities, we have

$$\mu^{-1}(\mu(j)) \in \mathcal{C}_{\mu(j)}(\mathcal{E}^{\mu}_j \cup \mu^{-1}(\mu(j))).$$

Furthermore, $\mathcal{E}'_j \subseteq \mathcal{E}^{\mu}_j$ and $\succsim$ being substitutable imply that

$$\mu^{-1}(\mu(j)) \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))).$$

Removing j from the choice set, we conclude, again using substitutability, that $\mu^{-1}(\mu(j))\backslash\{j\}$ is a subset of a chosen element from $\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}$. In other words

$$\mu^{-1}(\mu(j))\backslash\{j\} \subseteq S' \quad \text{for some} \quad S' \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}).$$

Any such S' has exactly one element from $\mathcal{E}'_j$, and let E'_j be the set of those elements:

$$E'_j = \left\{ \ell \;\middle|\; \begin{array}{c} \ell \in \mathcal{E}'_j, \quad \text{and} \quad (\mu^{-1}(\mu(j))\backslash\{j\}) \cup \{\ell\} = S' \\ \text{for some } S' \in \mathcal{C}_{\mu(j)}(\mathcal{E}'_j \cup \mu^{-1}(\mu(j))\backslash\{j\}) \end{array} \right\}$$

Thus, E'_j is a nonempty subset of N' for each $j \in N'$. Consider a directed graph whose set of vertices is N'. For each $i \in E'_j$, let there be a directed edge from i to j. Therefore, every vertex in this graph has an incoming edge, and since it is a finite graph, there must be a cycle.

Let the shortest cycle in this graph consist of students $i_0, i_1, \ldots, i_{n-1}, i_n = i_0$, where $n \geq 2$, and there is an edge from i_ℓ to $i_{\ell+1}$ for $\ell = 0, 1, \ldots, n-1$. Denoting $\mu(i_\ell) = x_\ell$, since i_ℓ envy $i_{\ell+1}$, we have $x_\ell \neq x_{\ell+1}$ for each ℓ. In fact, these schools $x_0, \ldots, x_{n-1}$ must be distinct, for otherwise we would have a shorter cycle, which would give a Pareto improvement over μ, involving a smaller number of students improving. To be more precise, if $x_0 = x_k$ for some $k \leq n-1$, then the cyclic trade which allows i_ℓ take $x_{\ell+1}$ for $\ell = 0, \ldots, k-1$, and letting i_k take x_0 would lead to a Pareto improvement over μ. Since $k < n$, this would contradict with the assumption that ν was the "smallest" improvement over μ. Since $\mu(i_\ell) = x_\ell$, the students $i_0, \ldots, i_{n-1}$ are necessarily distinct.

The fact that μ respects priorities implies that it is non-wasteful. Since each x_ℓ is desired by some student at assignment μ, all seats at these schools must be assigned

[10]Lemma 1 of Erdil and Ergin (2008)

under μ. Denoting $S_{x_\ell} = \mu^{-1}(x_\ell)\backslash\{i_\ell\}$, we know that $S_{x_0}, \ldots, S_{x_{n-1}}$ are mutually disjoint subsets of $N \setminus \{i_0, i_1, \ldots, i_{n-1}\}$, because $x_0, x_1, \ldots, x_{n-1}$ are distinct schools. Moreover we have

(1) $|S_{x_\ell}| = q_{x_\ell} - 1$,

(2)
$$\begin{aligned} & i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\ & i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\ & \quad \vdots \\ & i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}) \\ & i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\}) \end{aligned} \tag{*}$$

because otherwise, if student i_ℓ were to be in $DC_{x_{\ell+1}}(S_{x_{\ell+1}} \cup \{i_{\ell+1}, i_\ell\})$ for some ℓ, then we would have $S_{x_{\ell+1}} \cup \{i_{\ell+1}\} \notin \mathcal{C}_{x_{\ell+1}}(S_{x_{\ell+1}} \cup \{i_{\ell+1}, i_\ell\})$, contradicting stability of μ.

Let ω be the assignment derived from μ by letting the students $i_0, i_1, \ldots, i_{n-1}$ exchange their schools along the improvement cycle suggested above. In other words,

$$\omega(i) = \begin{cases} \mu(i) & i \neq i_\ell \\ \mu(i_{\ell+1}) & i = i_\ell \end{cases}$$

ω Pareto dominates μ, whereas μ is constrained efficient, so ω must not be stable. Therefore the cyclic trade letting i_ℓ take $\mu(i_{\ell+1})$ for $\ell = 0, 1, \ldots, n-1, n \equiv 0$ cannot be respecting priorities. Then we know from Proposition 1 that there must be a blocking pair involving one of these schools. Suppose that j and x_0 form a blocking pair for ω, i.e., $\omega^{-1}(x_0) \notin \mathcal{C}_x(\omega^{-1}(x_0) \cup \{j\})$. Then $x_0 P_j \omega(j)$ and

$$j \in DC_{x_0}(\omega^{-1}(x_0) \cup \{j\}) = DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}). \tag{**}$$

First, note that $j \neq i_{n-1}$, because $\omega(i_{n-1}) = x_0 P_j \omega(j)$. Secondly, $j \neq i_0$, because $\omega(i_0) P_{i_0} \mu(i_0) = x_0$, while $x_0 P_j \omega(j)$. And lastly if $j = i_k$ for some $1 \leq k \leq n-2\}$, then we have an envy cycle

$$i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_k \rightarrow i_0$$

which would allow a Pareto improvement involving only $k+1 \leq n-1$ students, contradicting our earlier choice of a smallest Pareto improvement over μ. Thus $j \notin \{i_0, \ldots, i_{n-1}\}$.

Furthermore, stability of μ implies

$$j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}). \tag{***}$$

Thus, combining (*), (**), and (***), we have a generalized weak cycle

$$\begin{aligned}
& j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}) \\
& j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}) \\
& i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\
& i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\
& \quad \vdots \\
& i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}) \\
& i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\})
\end{aligned}$$

with $|S_{x_\ell}| = q_{x_\ell} - 1$ for $\ell = 0, 1, \ldots, n-1$.

Step 2: If $\succsim$ has a generalized weak cycle, then it has a weak cycle.

Proof of Step 2: Suppose that $\succsim$ has a generalized weak cycle and let the size of its shortest generalized weak cycle be n. We will show that $n = 2$, which will prove step 2, since a weak cycle is a generalized weak cycle of size 2. Suppose that $x_0, x_1, \ldots, x_{n-1} \in X$; $j, i_0, i_1, \ldots, i_{n-1} \in N$ and $S_{x_0}, \ldots, S_{x_{n-1}} \subseteq N \setminus \{j, i_0, \ldots, i_{n-1}\}$ form a shortest generalized weak cycle. We will assume that it is of size $n \geq 3$, and reach a contradiction.

Let us look at the the set of definitely chosen students from $S_{x_1} \cup \{i_0, i_2\}$ according to the priorities of x_1. Is i_0 in this set or not?

If so, i.e., if $i_0 \in DC_{x_1}(S_{x_1} \cup \{i_0, i_2\})$, then

$$\begin{aligned}
& i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_0\}) \\
& i_0 \in DC_{x_1}(S_{x_1} \cup \{i_0, i_2\}) \\
& i_2 \notin DC_{x_1}(S_{x_1} \cup \{i_1, i_2\}) \\
& i_1 \notin DC_{x_2}(S_{x_2} \cup \{i_2, i_1\}),
\end{aligned}$$

which is a weak cycle, i.e., a generalized weak cycle of length 2, contradicting with our assumption of shortest cycle being of length at least 3.

If on the other hand, $i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_0, i_2\})$, then we get the following generalized weak cycle

$$\begin{aligned}
&j \notin DC_{x_0}(S_{x_0} \cup \{i_0, j\}) \\
&j \in DC_{x_0}(S_{x_0} \cup \{i_{n-1}, j\}) \\
&i_{n-1} \notin DC_{x_0}(S_{x_0} \cup \{i_0, i_{n-1}\}) \\
&i_{n-2} \notin DC_{x_{n-1}}(S_{x_{n-1}} \cup \{i_{n-1}, i_{n-2}\}) \\
&\quad \vdots \\
&i_2 \notin DC_{x_3}(S_{x_3} \cup \{i_3, i_2\}) \\
&i_0 \notin DC_{x_1}(S_{x_1} \cup \{i_2, i_0\})
\end{aligned}$$

with $|S_{x_\ell}| = q_{x_\ell} - 1$ for $\ell = 0, 1, 3, \ldots, n-1$. This cycle is shorter than the one we started with, because it does not have x_2, hence yields the desired contradiction to our original cycle being the shortest.

$\underline{\Longrightarrow}$**:** Let N, X, and q and $\succsim$ be given. Assume that $\succsim$ has a weak cycle. Let $i, j, k \in N$, and $x, y \in X$ such that there exist $S_x, S_y \subseteq N \backslash \{i, j, k\}$ with $S_x \cap S_y = \varnothing$ satisfying

$$\begin{aligned}
&j \notin DC_x(S_x \cup \{i, j\}) \\
&j \in DC_x(S_x \cup \{k, j\}) \\
&k \notin DC_x(S_x \cup \{k, i\}) \\
&i \notin DC_y(S_y \cup \{k, i\})
\end{aligned}$$

with $|S_x| = q_x - 1$ and $|S_y| = q_y - 1$.

Consider the preference profile R where students in S_x and S_y, respectively, rank x and y as their top choice, and the preferences of i, j, and k are such that $yP_i xP_i iP_i \cdots$, $xP_j jP_j \cdots$, and $xP_k yP_k kP_k \cdots$. Finally, let students outside $S_x \cup S_y \cup \{i, j, k\}$ prefer not to be assigned to any school. Consider the assignment μ such that for each $\ell \in S_x \cup \{i\}$ one has $\mu(\ell) = x$, and for each $\ell \in S_y \cup \{k\}$ one has $\mu(\ell) = y$. Now the only candidates for blocking pairs are $(j, x), (k, x)$, and (i, y). However, the weak cycle conditions are such that $j \notin DC_x(S_x \cup \{i, j\})$ $k \notin DC_x(S_x \cup \{k, i\})$, and $i \notin DC_y(S_y \cup \{k, i\})$, ensuring that μ respects priorities $\succsim$. Moreover, there is only one assignment that Pareto dominates μ, namely the assignment ν obtained from μ by letting i and k trade their assigned schools. Since $j \in DC_x(S_x \cup \{j, k\})$, $xP_j\nu(j)$ and $\nu^{-1}(x) = S_x \cup \{k\}$, the assignment ν does not respect $\succsim$. Thus μ is constrained efficient, but not Pareto efficient. $\square$

A.4 Proof of Proposition 4

We will now show that if a stable assignment μ is Pareto dominated by another stable assignment ν, then μ must admit a SIC. From this, it will follow that if μ does not admit a SIC, then it must be constrained efficient.

Let $N' = \{i \in N \mid \mu(i) \neq \nu(i)\}$ and $X' = \{\nu(i) \mid i \in N'\}$. For any $i \in N'$, we know by the reshuffling lemma that $\mu(i) \in X'$.

Let $\mu(i) = x$. Denote

$$D_x^\mu = \{j \in N \mid xP_j\mu(j)\}, \quad D_x' = \{j \in N' \mid xP_j\mu(j)\}, \quad D_x'' = \{j \in N\backslash N' \mid xP_j\mu(j)\}$$

and set

$$\bar{D}_x = D_x' \sqcup D_x'' \sqcup \mu^{-1}(x) = D_x^\mu \sqcup \mu^{-1}(x).$$

Stability of μ implies that

$$\mu^{-1}(x) \in \mathcal{C}_x(\bar{D}_x)$$

Moreover, stability of ν implies that

$$D_x'' \subseteq T'' \text{ for some } T'' \in \mathcal{R}_x(D_x^\nu \sqcup \nu^{-1}(x)) \qquad (\star).$$

ν Pareto dominates μ, so those who desire x at ν, desire x at μ as well. Therefore $D_x^\nu \subseteq D_x^\mu$. Moreover, if $j \in \nu^{-1}(x)$, then either $j \in \mu^{-1}(x)$ or $j \in D_x'$. And finally, since $\mu(i) = x$ and $i \in N'$, we know that $i \notin \nu^{-1}(x)$, and $\nu(i)P_ix$. Therefore $i \notin D_x^\nu$. Thus

$$D_x^\nu \sqcup \nu^{-1}(x) \subseteq D_x^\mu \cup \nu^{-1}(x) \subseteq \bar{D}_x\backslash\{i\}. \qquad (\star\star)$$

Now we conclude by using $(\star), (\star\star)$, and substitutability that

$$D_x'' \subseteq T' \text{ for some } T' \in \mathcal{R}_x(\bar{D}_x\backslash\{i\}).$$

Denoting

$$S' = (\bar{D}_x\backslash\{i\})\backslash T',$$

we have

$$S' \in \mathcal{C}_x(\bar{D}_x\backslash\{i\}) \quad \text{and} \quad S' \cap D_x'' = \varnothing.$$

Note that

$$\bar{D}_x\backslash\{i\} = D_x' \sqcup D_x'' \sqcup \mu^{-1}(x)\backslash\{i\},$$

and $|\mu^{-1}(x)\backslash\{i\}| \leq q_x - 1$. Since $\succsim$ is acceptant, and $|\bar{D}_x\backslash\{i\}| \geq q_x$, we must have $|S'| \geq q_x$. Because of $|\mu^{-1}(x)\backslash\{i\}| \leq q_x - 1$ and that $S' \cap D''_x = \varnothing$, we have

$$S' \cap D'_x \neq \varnothing.$$

Hence, there exists $i' \in D'_x$ such that $\{i'\} \cup \mu_x^{-1}\backslash\{i\} \in \mathcal{C}_x(\bar{D}_x\backslash\{i\})$, i.e.,

$$i' \in E_i^{\mu}.$$

Now construct a directed graph with N' being its set of vertices. For any $i \in N'$, the above argument shows that there is $i' \in N'$ such that $i \in E_i^{\mu}$, so draw an edge $i' \to i$. Since this is a finite graph with every vertex having an incoming edge, there must be cycle. By construction, this is a SIC. □

A.5 Proof of Proposition 5

Denote the assignment obtained by carrying out this SIC by ν, i.e., define matching ν as

$$\nu(j) = \begin{cases} \mu(i_{\ell+1}) & \text{if } j = i_\ell \\ \mu(j) & \text{otherwise} \end{cases}$$

Case 1: If the schools $\mu(i_0), \mu(i_1), \ldots, \mu(i_{n-1})$ are distinct, then it is "straightforwardly verified" that ν is stable.

Case 2: Now consider the case in which the schools $\mu(i_0), \mu(i_1), \ldots, \mu(i_{n-1})$ are not distinct. Suppose for a contradiction that ν is not stable. So by Proposition 1 it must admit a blocking pair (j, x), with $j \in N$, and $x \in X$. That is,

$$j \in DC_x(\nu^{-1}(x) \cup \{j\}).$$

Note that such a school x must appear more than once in the SIC, for otherwise $\nu^{-1}(x) = \mu^{-1}(x)\backslash\{i_{\ell+1}\} \cup \{i_\ell\}$ and $i_\ell \in E_{i_{\ell+1}}$, and hence $j \notin DC_x((\mu^{-1}(x)\backslash\{i_{\ell+1}\}) \cup \{i_\ell, j\})$, contradicting with (j, x) being a blocking pair.

Suppose that the school x is involved in moves $i_{k^t} \to i_{k^t+1}$ for $t = 1, \ldots, m$, so the SIC looks like:

$$i_0 \to \cdots \to i_{k^1} \to i_{k^1+1} \to \cdots \to i_{k^2} \to i_{k^2+1} \to \cdots \to i_{k^m} \to i_{k^m+1} \to \cdots \to i_{n-1},$$

where $k^t \in \{0, \ldots, n-1\}$ and $\mu(i_{k^t+1}) = x$ for all $t \in \{1, 2, \ldots, m\}$.

Since (j, x) is a blocking pair for ν, we have $xP_j\nu(j)$ and $j \in DC_x(\nu^{-1}(x) \cup \{j\})$. Thus $xP_j\nu(j)R_j\mu(j)$, and $j \in \mathcal{E}^{\mu}_{k^t+1}$ for all t.

The definition of SIC and substitutability implies that for each $t \in \{1, \ldots, m\}$ there exists A_t such that

$$[\mu^{-1}(x)\backslash\{i_{k^1+1}, \ldots i_{k^m+1}\}] \cup \{i_{k^t}\} \ \subseteq \ A_t \in \mathcal{C}_x(\nu^{-1}(x) \cup \{j\}),$$

because $i_{k^t} \in \mathcal{E}^{\mu}_{i_{k^t+1}}$ for all $t \in \{1, 2, \ldots, m\}$.

Note that j is in A_t, because $j \in DC_x(\nu^{-1}(x) \cup \{j\})$.

Thus we get

$$[\mu^{-1}(x)\backslash\{i_{k^1+1}, \ldots, i_{k^m+1}\}] \cup \{i_{k^t}\} \cup \{j\} \ \subseteq \ A_t \in \mathcal{C}_x(\nu^{-1}(x) \cup \{j\}).$$

Let us write A_t as the disjoint union

$$A_t = B_t \sqcup \mu^{-1}(x)\backslash\{i_{k^1+1}, \ldots, i_{k^m+1}\}$$

Note that, for all $t \in \{1, \ldots, m\}$:

$$\{i_{k^t}, j\} \subseteq B_t \subseteq \{i_{k^1}, \ldots, i_{k^m}, j\} \quad \text{and } |B_t| = m.$$

There must exist t, t' such that $B_t \neq B_{t'}$, for otherwise $\{i_{k^1}, \ldots, i_{k^m}, j\} \subseteq B_t$ contradicting with $|B_t| = m$. Let the symmetric difference of B_t and $B_{t'}$ be $\{i_{k^r}, i_{k^s}\}$, where $r < s$, so that

$$B_t = \tilde{B} \cup \{i_{k^r}\} \quad \text{and} \quad B_{t'} = \tilde{B} \cup \{i_{k^s}\},$$

and hence

$$A_t = \tilde{A} \cup \{i_{k^r}\} \quad \text{and} \quad A_{t'} = \tilde{A} \cup \{i_{k^s}\},$$

where $\tilde{A} = \tilde{B} \sqcup (\mu^{-1}(x)\backslash\{i_{k^1}, \ldots, i_{k^m}\})$.

Since

$$\begin{aligned}
&A_t, A_{t'} \in \mathcal{C}_x(\nu^{-1}(x) \cup \{j\}) \\
&\nu^{-1}(x) \cup \{j\} \subseteq \mathcal{E}^{\mu}_{i_{k^s+1}} \cup \mu^{-1}(x)\backslash\{i_{k^s+1}\} \text{ and} \\
&\mu^{-1}(x)\backslash\{i_{k^s+1}\} \cup \{i_{k^s}\} \in \mathcal{C}_x(\mathcal{E}^{\mu}_{i_{k^s+1}} \cup \mu^{-1}(x)\backslash\{i_{k^s+1}\}),
\end{aligned}$$

ETE[11] implies that $\mu^{-1}(x)\backslash\{i_{k^s+1}\} \cup \{i_{k^r}\} \in \mathcal{C}_x(\mathcal{E}^{\mu}_{i_{k^s+1}} \cup \mu^{-1}(x)\backslash\{i_{k^s+1}\})$, and therefore

$$i_{k^r} \in E^{\mu}_{i_{k^s+1}}.$$

Hence there is a shorter SIC which looks like

$$i_0 \to \cdots \to i_{k^1} \to i_{k^1+1} \to \cdots \to i_{k^r} \to i_{k^s+1} \to \cdots \to i_{k^m} \to i_{k^m+1} \to \cdots \to i_{n-1},$$

contradicting with the initial assumption that the original SIC was the shortest such cycle. □

A.6 Proof of Claim 1

It is obvious that $\succsim$ is acceptant by (RR) condition and $\succsim^{pre}$ which is defined over the set of students. We need to show that $\succsim$ is substitutable.

Lemma 1 *For every $\succsim$ constructed from $\succsim^{pre}$ satisfying (RC) and (RR) conditions, if $|S| \geq q_x$, then $S' \in \mathcal{C}(S)$ if and only if S' has the following properties:*[12]

(1) $|S'| = q_x$

(2) $|S_\tau| \leq q_\tau \Rightarrow S_\tau \subseteq S'$

(3) $|S_\tau| > q_\tau \Rightarrow |S'_\tau| \geq q_\tau$ *and for all* $s_i \in S'_\tau$, $s_i \succsim \hat{s}$ *for all* $\hat{s} \in S_\tau \backslash S'_\tau$

(4) $|S'_\tau| > q_\tau \Rightarrow$ *for all* $s_i \in S'_\tau$, $s_i \succsim \hat{s}$ *for all* $\hat{s} \in S \backslash S'$.

Proof:
<u>(⇒)</u> Let $S' \in \mathcal{C}(S)$ and fix arbitrary. Since $\succsim$ is acceptant and $|S| > q_x$, $|S'| = q_x$ for any $S' \in \mathcal{C}(S)$. Suppose $S' \in D_a$.

For (2), if $S' \in D_0$ then the condition trivially holds, so we assume $a \geq 1$ for D_a and suppose for a contradiction. Then there is $S_{\tau'}$ such that $S_{\tau'} \not\subseteq S'$. Pick an agent in $S_{\tau'} \backslash S'$, denoted by s'. Since $|S'| = q_x$ and $\sum_\tau q_\tau = q_x$, there must be τ'' such that $|S'_{\tau''}| > q_{\tau''}$. Let $s'' \in S'_{\tau''}$.

[11] Recall that ETE requires that if i_{k^r} can substitute i_{k^s} to complement some set A, then she can substitute him to complement any other set B in any larger applicants. See Footnote **??**.

[12] By construction, a priority ordering over single elements are the same between $\succsim$ and $\succsim^{pre}$, so we use $\succsim$ for the comparison among single elements.

Consider $S'' = S' \backslash \{s''\} \sqcup \{s'\}$. Since $S' \in D_a$,

$$\sum_{\tau} ||S'_\tau| - q_\tau| = \sum_{\tau \neq \tau', \tau''} ||S'_\tau| - q_\tau| + ||S'_{\tau'}| - q_{\tau'}| + ||S'_{\tau''}| - q_{\tau''}| = 2i.$$

On the other hand,

$$\sum_{\tau} ||S''_\tau| - q_\tau| = \sum_{\tau \neq \tau', \tau''} ||S''_\tau| - q_\tau| + ||S''_{\tau'}| - q_{\tau'}| + ||S''_{\tau''}| - q_{\tau''}|,$$

and

$$\begin{aligned} ||S'_{\tau'}| - q_{\tau'}| &= ||S''_{\tau'}| - q_{\tau'}| + 1 \\ ||S'_{\tau''}| - q_{\tau''}| &= ||S''_{\tau''}| - q_{\tau''}| + 1. \end{aligned}$$

Hence, $S'' \in D_{a-1}$ and by (RC), $S'' \succ S'$, but $S' \in \mathcal{C}(S)$, a contradiction.

For (3), suppose that $|S_\tau| \geq q_\tau$ and the condition does not hold. Then there is $\hat{s} \in S_\tau \backslash S'_\tau$ such that $\hat{s} \succ s'$ for some $s' \in S'_\tau$. Since they are in the same type, $S' \backslash \{s'\} \sqcup \{\hat{s}\} \in D_i$. From (RR), $S' \backslash \{s'\} \sqcup \{\hat{s}\} \succ S'$, a contradiction. Suppose $|S'_\tau| < q_\tau$, then by the similar argument to the proof of (2), we conclude that it never happens.

For (4), suppose not. Then there is $\hat{s} \in S \backslash S'$ such that $\hat{s} \succ s'$ for some $s' \in S'_\tau$. Since $\hat{s} \in S \backslash S'$ and (1) – (3), there is $S_{\tau'}$ such that $|S_{\tau'}| > q_{\tau'}$ and $\tau(\hat{s}) = \tau'$. This fact and $|S'_\tau \backslash \{s'\}| \geq q_\tau$ imply $S' \backslash \{s'\} \sqcup \{\hat{s}\} \in D_a$. Then (RR) implies $S' \backslash \{s'\} \sqcup \{\hat{s}\} \succ S'$, a contradiction. □

($\Leftarrow$) Suppose S' satisfies (1) – (4) but $S' \notin \mathcal{C}(S)$. Then there is $S'' \in \mathcal{C}(S)$ such that $S'' \succ S'$. Note that S'' satisfies (1) – (4) by ($\Rightarrow$) and S' is in D_a if and only if S'' is in D_a. By induction on $|S' \cap S''|$.

(step 1) $|S' \cap S''| = q_x - 1$. Then let $s'_1 \in S' \backslash S''$ and $s''_1 \in S'' \backslash S'$. Since $S', S'' \in D_a$ and by supposition,

$$S'' \succ S' \Rightarrow s''_1 \succ s'_1,$$

by (RR). Since $s''_1 \in S \backslash S'$ and S' satisfies (4),

$$|S_{\tau(s'_1)} \cap S'| = q_{\tau(s'_1)}.$$

Then $|S_{\tau(s'_1)} \cap S''| < q_{\tau(s'_1)}$, even though $|S_{\tau(s'_1)}| > q_{\tau(s'_1)}$. A contradiction.

(step n) Assume the conclusion holds for the case that $|S' \cap S''| \leq q_x - (n-1)$, and consider $|S' \cap S''| = q_x - n$. Let $s'_1, \ldots, s'_n \in S' \backslash S''$ and $s''_1, \ldots, s''_n \in S'' \backslash S'$. Without loss of generality, we assume that $s'_1 \succsim s'_2 \succsim \cdots \succsim s'_n$ and $s''_1 \succsim s''_2 \succsim \cdots \succsim s''_n$.

Case 1 ($s'_1 \succ s''_1$). Then $S'' \backslash \{s''_1\} \sqcup \{s'_1\} \in D_a$ and $S'' \backslash \{s''_1\} \sqcup \{s'_1\} \succ S''$, a contradiction.

Case 2 ($s'_1 \sim s''_1$). Then $S'' \sim S'' \backslash \{s''_1\} \sqcup \{s'_1\}$. This means that $S'' \backslash \{s''_1\} \sqcup \{s'_1\} \succ S'$ and $|[S'' \backslash \{s''_1\} \sqcup \{s'_1\}] \cap S'| = q_x - (n-1)$. This case reduces to $n-1$, and by assumption, the conclusion holds.

Case 3 ($s''_1 \succ s'_1$). Then $s''_1 \in S \backslash S'$ and (4) imply that $|S'_{\tau(s'_1)}| = q_{\tau(s'_1)}$. Note that $\tau(s'_1) \neq \tau(s''_1)$. Then

$$\begin{aligned} |S_{\tau(s'_1)} \cap S''| &< q_{\tau(s'_1)} \quad if \quad \tau(s''_i) \neq \tau(s'_1) \quad \forall i \in \{2, \ldots, n\} \\ |S_{\tau(s'_1)} \cap S''| &= q_{\tau(s'_1)} \quad if \quad \tau(s''_i) = \tau(s'_1) \quad \exists i \in \{2, \ldots, n\} \end{aligned}$$

Clearly, a case that $|S_{\tau(s'_1)} \cap S''| < q_{\tau(s'_1)}$ leads to a contradiction. When $|S_{\tau(s'_1)} \cap S''| = q_{\tau(s'_1)}$, since $|S_{\tau(s'_1)}| > q_{\tau(s'_1)}$, $s'_1 \succsim s''_i$. Then $S'' \backslash \{s''_i\} \sqcup \{s'_1\} \in D_a$, and

$$S'' \backslash \{s''_i\} \sqcup \{s'_1\} \succsim S''.$$

Since $S'' \in \mathcal{C}(S)$, it must be

$$S'' \backslash \{s''_i\} \sqcup \{s'_1\} \sim S''.$$

Then $s'_1 \sim s''_i$. Therefore $S'' \backslash \{s''_i\} \sqcup \{s'_1\} \in \mathcal{C}(S)$ and $S'' \backslash \{s''_i\} \sqcup \{s'_1\} \succ S'$. Notice that $|[S'' \backslash \{s''_i\} \sqcup \{s'_1\}] \cap S'| = q_x - (n-1)$, which reduces to $n-1$. □

Proof of Claim 1:

Without loss of generality, we just focus on $S \sqcup \{s''\}$ and S. Since $\succsim$ is acceptant, it suffices to show a case that $|S| \geq q_x + 1$.

Proof of the condition (a) Suppose $S' \in \mathcal{C}(S \sqcup \{s''\})$ and $S' \in D_a$. If $s'' \notin S'$, then $S' \in \mathcal{C}(S)$ and the condition (a) trivially holds. So we assume that $s'' \in S'$.

Case 1 $|S_{\tau(s'')}| \leq q_{\tau(s'')}$. Then since $S_{\tau(s'')} \subseteq S'$, for all $\hat{s} \in S \backslash S'$,

$$S' \backslash \{s''\} \sqcup \{\hat{s}\} \in D_{a+1}.$$

Then we have $\hat{s}_1 \succsim \hat{s}_2 \succsim \cdots \succsim \hat{s}_n$ for $S \backslash S'$ by (RR).

We claim that $S'' = S' \backslash \{s''\} \sqcup \{\hat{s}_1\} \in \mathcal{C}(S)$. Clearly $|S''| = q_x$.

For all S''_τ with $|S''_\tau| \leq q_\tau$, $S''_\tau = S'_\tau \subseteq S''$ if $\tau \neq \tau(s'')$. $S''_{\tau(s'')} = S'_{\tau(s'')} \backslash \{s''\} \subseteq S''$.

For all S''_τ with $|S''_\tau| > q_\tau$, since $\hat{s}_1 \succsim \hat{s}$ for all $\hat{s} \in S \backslash S'$ and

$$q_{\tau(\hat{s}_1)} \leq |S'_{\tau(\hat{s}_1)}| < |S''_{\tau(\hat{s}_1)}|,$$

(3) and (4) holds. By the lemma 1, $S'' \in \mathcal{C}(S)$.

Case 2 $|S_{\tau(s'')}| > q_\tau$. If $|S'_{\tau(s'')}| = q_{\tau(s'')}$, then there is $\hat{s} \in S \backslash S'$ such that $\tau(\hat{s}) = \tau(s'')$. For these $\hat{s}$, $S' \backslash \{s''\} \sqcup \{\hat{s}\} \in D_a$. By (RR), we can find $\hat{s}_1$ who is at least as good as any other agent who is in $S_{\tau(s'')}$. It is easy to see that $S' \backslash \{s''\} \sqcup \{\hat{s}_1\}$ satisfies (1) – (4).

Otherwise, $|S'_{\tau(s'')}| > q_{\tau(s'')}$. Then for all $\hat{s} \in S \backslash S'$,

$$S' \backslash \{s''\} \sqcup \{\hat{s}\} \in D_a.$$

Then we have $\hat{s}_1 \succsim \hat{s}_2 \succsim \cdots \succsim \hat{s}_n$ for $S \backslash S'$ by (RR). It is analogous to see that $S' \backslash \{s''\} \sqcup \{\hat{s}_1\}$ satisfies (1) – (4).

Hence, the condition (a) holds. □

Proof of the condition (b) Suppose $R' \in \mathcal{R}(S)$. By definition, $R' = S \backslash S'$ for some $S' \in \mathcal{C}(S)$. We claim that there is $\hat{s} \in (S \sqcup \{s''\}) \backslash R'$ such that $R' \sqcup \{\hat{s}\} \in \mathcal{R}(S \sqcup s'')$.

Case 1 $|(S \sqcup \{s''\})_{\tau(s'')}| \leq q_{\tau(s'')}$. Then $|S_{\tau(s'')}| < q_{\tau(s'')}$. This implies that there is τ' such that $|S'_{\tau'}| > q_{\tau'}$. Then for all $s_i \in R'_{\tau'}$, $\hat{s} \succsim s_i$ for all $\hat{s} \in S'_{\tau'}$. Consider such types $\{\tau'_1, \cdots, \tau'_m\}$. Take $s^* \in \bigcup_{i \in \{1,\ldots,m\}} S_{\tau'_i}$ in a way that $\hat{s} \succsim s^*$ for all $\hat{s} \in \bigcup_{i \in \{1,\ldots,m\}} S_{\tau'_i}$. We see that $R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{s''\})$. Let $S'' = (S \sqcup \{s''\}) \backslash (R' \sqcup \{s^*\})$. Since $|(S \sqcup \{s''\})_{\tau(s'')}| \leq q_{\tau(s'')}$, $(S \sqcup \{s''\})_{\tau(s'')} \subseteq S''$. For $\tau(s^*)$, $|S''_{\tau(s^*)}| \geq q_{\tau(s^*)}$ and by construction, S'' satisfies other properties. Hence, $S'' \in \mathcal{C}(S \sqcup \{s''\}) \Leftrightarrow R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{s''\})$.

Case 2 $|(S \sqcup s'')_{\tau(s'')}| > q_{\tau(s'')}$. Then we can find $\hat{s} \in (S \sqcup \{s''\})_{\tau(s'')}$ such that $s_i \succsim \hat{s}$ for all $s_i \in (S \sqcup \{s''\})_{\tau(s'')}$. Consider τ' such that $|S'_{\tau'}| > q_{\tau'}$ and let them be $\{\tau'_1, \dots, \tau'_m\}$ (possibly empty). If for all such τ'_i, $s \succsim \hat{s}$, for all $s \in S'_{\tau'_i}$ or there is no such τ_i, then $S \backslash (R' \sqcup \{\hat{s}\})$ satisfies (1) – (4) and we are done. Otherwise there is $s_i \in S'_{\tau'_i}$ such that $\hat{s} \succ s_i$ for some $i \in \{1, \dots, m\}$. Then we can find $\hat{s}_i$ such that $s_i \succsim \hat{s}_i$ for all $s_i \in S'_{\tau'_i}$. Let s^* be such that $\hat{s}_i \succsim s^*$ for any i. Then it also easy to see that $S'' = S \backslash (R' \sqcup \{s^*\})$ satisfies (1) – (4). Therefore, $R' \sqcup \{s^*\} \in \mathcal{R}(S \sqcup \{s''\})$.

Hence the condition 2 holds. □

A.7 Proof of Claim 2

We only need to show that if a stable assignment admits a SIC, then a new assignment followed by the SIC is stable.

As the same as the proof of Proposition 4, consider the shortest SIC and denote the assignment obtained by carrying out the SIC by ν. First of all, if the schools involved in the SIC, $\mu(i_0), \dots, \mu(i_{n-1})$, are distinct, it is obvious that ν is stable.

We assume that the schools are not distinct. Suppose for a contradiction, that ν is not stable. Suppose that there is a blocking pair (j, x) and the school x is involved in moves $i_{k^t} \to i_{k^t+1}$ for $t = 1, \dots, m$.

Note that $\tau(i_{k^t}) \neq \tau(i_{k^u})$, for any $t, u \in \{1, \dots, m\}$ and $t \neq u$. Otherwise $\tau = \tau(i_{k^t}) = \tau(i_{k^u})$ for some t and $u \neq t$. Then we can see that $i_{k^u} \in E^{\mu}_{i_{k^t+1}}$.

Since $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$ and $i_{k^u} \in \mathcal{E}^{\mu}_{i_{k^t+1}}$,

$$\mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^t}\} \succsim \mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^u}\}.$$

Since i_{k^t} and i_{k^u} are in the same type,

$$\mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^t}\} \in D_a$$
$$\mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^u}\} \in D_a,$$

which implies by (RR) that

$$i_{k^t} \succsim i_{k^u}.$$

Similarly, $i_{k^u} \succsim i_{k^t}$, and therefore

$$i_{k^t} \sim i_{k^u}.$$

This implies $i_{k^u} \in E^{\mu}_{k^u+1}$, which is contradicting with the supposition that the SIC is the shortest.

Consider a blocking pair (j, x) such that $j \in DC_x(\nu^{-1} \cup \{j\})$.

Case 1 $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(j)}| \leq q_{\tau(j)}$. Since μ is stable and j is not in $\mu^{-1}(x)$, $|(\mu^{-1}(x))_{\tau(j)}| \geq q_{\tau(j)}$. If $|(\mu^{-1}(x))_{\tau(j)}| = q_{\tau(j)}$, then there is i_{k^t+1} such that $\tau(i_{k^t+1}) = \tau(j)$. Then $|\mu^{-1}(x) \backslash \{i_{k^t+1}\}| = q_{\tau(j)} - 1$. Since $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$, $j \in E^{\mu}_{i_{k^t+1}}$ and $\tau(i_{k^t}) = \tau(j)$. Since $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(j)}| \leq q_{\tau(j)}$, there must be another i_{k^u+1} such that $\tau(i_{k^u+1}) = \tau(j)$. If $|(\mu^{-1}(x))_{\tau(j)}| > q_{\tau(j)}$, then we can easily see that there are distinct i_{k^t+1}, i_{k^u+1} such that $\tau(i_{k^t+1}) = \tau(i_{k^u+1}) = \tau(j)$.

Therefore, there are at least two agents i_{k^t+1}, i_{k^u+1} such that $\tau(i_{k^t+1}) = \tau(i_{k^u+1}) = \tau(j)$ in the SIC. Then

$$E^{\mu}_{i_{k^t+1}} = E^{\mu}_{i_{k^u+1}},$$

which contradicts that the SIC is the shortest.

Case 2 $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(j)}| > q_{\tau(j)}$. Since $j \in DC_x(\nu^{-1}(x) \sqcup \{j\})$ and $|\nu^{-1}(x) \sqcup \{j\}| = q_x + 1$, there is i_{k^t} such that $i_{k^t} \notin A \in \mathcal{C}_x(\nu^{-1}(x) \sqcup \{j\})$. Then $|(\nu^{-1}(x) \sqcup \{j\})_{\tau(i_{k^t})}| > q_{\tau(i_{k^t})}$. Note that

$$j \succsim i_{k^t}$$

because if $\tau(i_{k^t}) = \tau(j)$, then $j \succsim i_{k^t}$ by (RR), and otherwise since $\mathcal{C}_x$ satisfies substitutability and $\nu^{-1}(x) \backslash \{i_{k^t}\} \sqcup \{j\} \in \mathcal{C}_x(\nu^{-1}(x) \sqcup \{j\})$, (4) of the lemma 1 implies $j \succsim i_{k^t}$.

Suppose $j \sim i_{k^t}$. Since $\nu^{-1}(x) \backslash \{i_{k^t}\} \sqcup \{j\}, \nu^{-1}(x) \in D_a$ for some a,

$$\nu^{-1}(x) \backslash \{i_{k^t}\} \sqcup \{j\} \sim \nu^{-1}(x),$$

implying that $j \notin DC_x(\nu^{-1} \sqcup \{j\})$.

Suppose $j \succ i_{k^t}$. We know that $\tau(i_{k^t}) \neq \tau(i_{k^u})$ for all $t, u \neq t \in \{1, \ldots, m\}$.

$$\begin{aligned} \left|(\mu^{-1}(x) \backslash \{i_{k^t+1}\} \sqcup \{i_{k^t}, j\})_{\tau(i_{k^t})}\right| &> \left|(\mu^{-1} \backslash \{i_{k^1+1}, \ldots, i_{k^m+1}\} \sqcup \{i_{k^t}, j\})_{\tau(i_{k^t})}\right| \\ &= \left|(\nu^{-1}(x) \sqcup \{j\})_{\tau(i_{k^t})}\right| > q_{\tau(i_{k^t})} \end{aligned}$$

Since $i_{k^t} \in E^{\mu}_{i_{k^t+1}}$, $i_{k^t} \succsim j$, a contradiction. $\square$

B Remarks

Remark 1 When $\mathcal{C}_x$ is correspondence, the conditions (a) and (b) in Definition 1 are not necessarily equivalent. We offer two priority rankings:

- $\mathcal{C}_x$ satisfies (a) but not (b): Suppose there are four students $\{i_1, i_2, i_3, i_4\}$ and a school x has two seats. A school x has the following priority ranking.

$\mathcal{C}_x$
$\{i_1, i_4\}$
$\{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\}, \{i_2, i_4\}, \{i_3, i_4\}$

  This priority ranking does not satisfy the condition (b) because

$$\mathcal{R}_x(\{i_1, i_2, i_3\}) = \{\{i_1\}, \{i_2\}, \{i_3\}\},$$

  but

$$\mathcal{R}_x(\{i_1, i_2, i_3, i_4\}) = \{\{i_2, i_3\}\},$$

  hence

$$\{i_1\} \not\subseteq \{i_2, i_3\}.$$

- $\mathcal{C}_x$ satisfies (b) but note (a): Suppose there are five students $\{i_1, i_2, i_3, i_4, i_5\}$ and a school x has two seats. A school x has the following priority ranking.

$\mathcal{C}_x$
$\{i_1, i_2\}, \{i_3, i_4\}$
$\{i_1, i_3\}, \{i_1, i_4\}, \{i_1, i_5\}, \{i_2, i_3\}, \{i_2, i_4\}$
$\{i_2, i_5\}, \{i_3, i_5\}, \{i_4, i_5\}$

  This does not satisfy (a) because

$$\mathcal{C}_x(\{i_1, i_2, i_3, i_4\}) = \{\{i_1, i_2\}, \{i_3, i_4\}\},$$

  but

$$\mathcal{C}_x(\{i_1, i_2, i_3\}) = \{\{i_1, i_2\}\},$$

  hence

$$\{i_3\} = \{i_3, i_4\} \cap \{i_1, i_2, i_3\} \not\subseteq \{i_1, i_2\}.$$

Remark 2 In an even more general formulation of priorities which allows *any* type of ties between sets of students, a stability-preserving Pareto improvement does not necessarily follow from cyclical trades students, in which each individual move preserves stability. Even if there is a cycle of students $i_1 \to i_2 \to \cdots \to i_m \to i_1$ such that each student can replace the next one without violating stability, the cycle does not necessarily preserve stability.[13] In order to see this, let $N = \{i_1, i_2, i_3, i_4, i_5\}$. Suppose that we have two schools x and y with $q_x = 2$, q_y=2. Students' preferences are:

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
x	y	x	y	x
y	x	y	x	

And the priorities are:

$\succsim_x$	$\succsim_y$
$\{i_2, i_4\}$	$\{i_1, i_2\}, \{i_1, i_3\}, \{i_3, i_4\}$
$\{i_1, i_4\}, \{i_2, i_3\}, \{i_2, i_5\}, \{i_4, i_5\}$	$\{i_1, i_4\}, \{i_2, i_3\}, \{i_3, i_4\}$
$\{i_1, i_3\}, \{i_3, i_5\}$	the rest
$\{i_1, i_2\}, \{i_1, i_5\}, \{i_3, i_4\}$	

The priority structure is acceptant and substitutable.[14] Let μ be

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ y & x & y & x & \varnothing \end{pmatrix}$$

One can verify that μ is stable. Consider the following replacement cycle

$$i_1 \to i_2 \to i_3 \to i_4 \to i_1,$$

in which each student can replace the next, because

$$\{i_1, i_4\} \in \mathcal{C}_x(\{i_1, i_3, i_4, i_5\}) \Longrightarrow i_1 \in E^{\mu}_{i_2}$$
$$\{i_1, i_2\} \in \mathcal{C}_y(\{i_1, i_2, i_4\}) \Longrightarrow i_2 \in E^{\mu}_{i_3}$$
$$\{i_2, i_3\} \in \mathcal{C}_x(\{i_1, i_2, i_3, i_5\}) \Longrightarrow i_3 \in E^{\mu}_{i_4}$$
$$\{i_3, i_4\} \in \mathcal{C}_y(\{i_2, i_3, i_4\}) \Longrightarrow i_4 \in E^{\mu}_{i_1}.$$

[13] This is in contrast with the case of responsive priorities studied in Erdil and Ergin (2008).

[14] The crucial point is that $\mathcal{C}_x(\{i_1, i_3, i_5\}) = \{\{i_1, i_5\}, \{i_3, i_5\}\}$. Without ties, the fact that $\{i_1, i_3\}$ is not chosen leads to a violation of substitutability, so the only if part is true in a strict priority structure.

Now construct ν by carrying out the above cycle:

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & y & x & y & \varnothing \end{pmatrix}$$

Clearly, ν Pareto dominates μ.

However, ν is not stable, because $\mathcal{C}_x(i_1, i_3, i_5) = \{\{i_1, i_5\}, \{i_3, i_5\}\}$, so (x, i_5) blocks ν.

In fact, μ is constrained efficient. Note that the only Pareto improvement over μ are the following:

$$\mu_1 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & y & y & x & \varnothing \end{pmatrix}$$

$\mathcal{C}_y(\{i_2, i_3, i_4\}) = \{i_3, i_4\}$, so (b, i_4) blocks μ_1.

$$\mu_2 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & x & y & y & \varnothing \end{pmatrix}$$

$\mathcal{C}_x(\{i_1, i_2, i_3\}) = \{i_2, i_3\}$, so (x, i_3) blocks μ_2.

$$\mu_3 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ y & y & x & x & \varnothing \end{pmatrix}$$

$\mathcal{C}_x(\{i_1, i_3, i_4\}) = \{i_1, i_4\}$, so (x, i_1) blocks μ_3.

$$\mu_4 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ y & x & x & y & \varnothing \end{pmatrix}$$

$\mathcal{C}_y(\{i_1, i_2, i_4\}) = \{i_1, i_2\}$, so (y, i_2) blocks μ_4.

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ x & y & x & y & \varnothing \end{pmatrix}$$

ν is not stable as pointed out before. Hence, there is no stable assignment which Pareto dominates μ, and so it is constrained efficient. $\Diamond$

Remark 3 In a general environment, the approach in Erdil and Ergin (2008) does not work out. Let x, y, z, w be distinct schools with $q_x = q_y = 2$ and $q_z = q_w = 1$. We have six students: $i, j, k_i, k_j, \ell_i, \ell_j$ and an assignment μ:

$$\mu = \begin{pmatrix} i & j & k_i & k_j & \ell_i & \ell_j \\ x & x & y & y & w & z \end{pmatrix}.$$

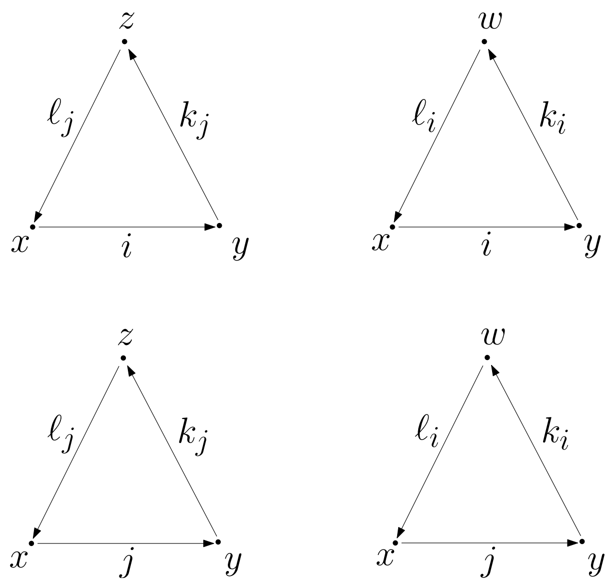

Figure 2: These cycles of schools are the only candidates for stable improvement cycles in the sense of Erdil and Ergin (2008).

The priority structure is such that

$$\mathcal{C}_x(\ell_i, \ell_j, i, j) = \{i, j\} \qquad \mathcal{C}_x(\ell_i, \ell_j, j) = \{\ell_i, j\} \qquad \mathcal{C}_x(\ell_i, \ell_j, i) = \{\ell_j, i\}$$

and

$$\mathcal{C}_y(k_i, k_j, i, j) = \{k_i, k_j\} \qquad \mathcal{C}_y(i, j, k_i) = \{i, k_i\} \qquad \mathcal{C}_y(i, j, k_j) = \{j, k_j\}$$

Suppose the preferences R are as below, where the boxes indicate the respective students' assignments under μ:

R_i	R_j	R_{k_i}	R_{k_j}	R_{ℓ_i}	R_{ℓ_j}
y	y	w	z	x	x
$\boxed{x}$	$\boxed{x}$	$\boxed{y}$	$\boxed{y}$	$\boxed{w}$	$\boxed{z}$

With these preferences, μ respects the priorities $\succsim$, but obviously, letting each student get their most preferred school preserves stability, while Pareto improving over μ. Can this improvement be achieved through Erdil and Ergin's stable improvement cycles? The only "candidates" for such cycles are shown in Figure 2.

However, none of these cycles preserves stability. The first one fails, because $\mathcal{C}_x(\ell_i, \ell_j, j) = \{\ell_i, j\}$, and thus ℓ_j cannot replace[15] i at school x. The second cycle fails to preserve

[15]Given a stable assignment, let us say that a student i can replace another student i' if the former

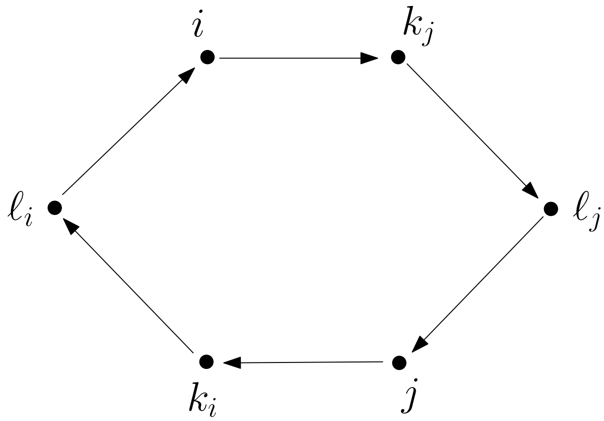

Figure 3: A stable student improvement cycle, in which the notation now indicates that i replaces k_j, and so on.

stability, because $\mathcal{C}_y(i, j, k_j) = \{j, k_j\}$, and therefore i cannot replace k_i. Likewise, the third one fails, because j cannot replace k_j at y, while for the fourth one ℓ_i cannot replace j at x.

Yet all of the students can move to their favorite schools while preserving stability. Thus, we can express a stability preserving student improving cycle illustrated in Figure 3 as a cycle of students (as opposed to a cycle of schools[16]as in Erdil and Ergin, 2008):

$$i \longrightarrow k_j \longrightarrow \ell_j \longrightarrow j \longrightarrow k_i \longrightarrow \ell_i \longrightarrow i$$

$\diamondsuit$

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replaces the latter in the set of students who receive x, the new set is a chosen set from among those who weakly desire x except the student i'.

[16]Note that this example does not rely on ties.

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